On Right-Angled Polygons in Hyperbolic Space

joint work with Edoardo Dotti

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(Oriented) Right-Angled Polygon

Definition

- finite sequence $(S_0, S_1, \ldots, S_{p-1})$ of (oriented) geodesics in $\mathbf{H}^n$
- $S_i \perp S_{i+1}$ and $S_{i-1} \neq S_{i+1}$
  considering $i \mod p$

Attention

in general *not* planar

Motivation

Delgove & Retailleau (2014)

- right-angled hexagons in $\mathbf{H}^5$
- upper half space model based on quaternions $\mathbb{H}$

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Dotti & D (2018)

- right-angled *p*-gons in $\mathbf{H}^{p-1}$
- upper half space model based on Clifford vectors $\mathbb{V}^{p-2}$

Overview

1. Clifford Algebra
2. Upper Half Space Model
3. Cross Ratio
4. Constructing Right-Angled Polygons

Clifford Algebra $\mathcal{C}_n$

Definition

$$\mathcal{C}_n := \langle i_1, \ldots, i_n \mid \forall j \neq k : i_j\, i_k = -i_k\, i_j, i_j^2 = -1 \rangle$$

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Examples

$$\mathcal{C}_0 = \mathbb{R}$$
$$\mathcal{C}_1 = \mathbb{C}$$
$$\mathcal{C}_2 = \mathbb{H}$$

Clifford Algebra $\mathcal{C}_n$

$$\mathcal{C}_n \ni x = \sum_I x_I \, I$$

with $x_I \in \mathbb{R}$, where the sum ranges over products

$$I = i_{k_1} \cdots i_{k_l}$$

with $1 \leq k_1 < \cdots < k_l \leq n$

$$\therefore \dim_{\mathbb{R}} \mathcal{C}_n = 2^n$$

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Three Involutions

- $\cdot^\star : i_{k_1} \cdots i_{k_m} \mapsto i_{k_m} \cdots i_{k_1}$
  antiautomorphism
- $\cdot' : i_{k_1} \cdots i_{k_m} \mapsto (-1)^m \, i_{k_1} \cdots i_{k_m}$
  automorphism
- $\bar{\cdot} = (\cdot')^\star = (\cdot^\star)'$
  antiautomorphism

Clifford Vectors $\mathbb{V}^{n+1}$

Definition

$$\mathcal{C}_n \supset \mathbb{V}^{n+1} := \left\{ x = x_0 + \sum_{j=1}^{n} x_j \, i_j \mid x_j \in \mathbb{R} \right\}$$

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- real part $\Re(x) := x_0$
- norm $|x|^2 = x\,\bar{x} = \sum_{j=0}^{n} x_j^2$
- invertible with $x^{-1} = \bar{x}/|x|^2$

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Clifford group Γ_n

group generated by all non-zero Clifford vectors

Square Root of a Clifford Vector

Square Root

For $x \in \mathbb{V}^{n+1} \setminus \mathbb{R}_{\leq 0}$ define

$$\sqrt{x} := \frac{|x| + x}{\sqrt{2\ (\Re(x) + |x|)}} \in \mathbb{V}^{n+1}.$$

$\pm\sqrt{x}$ are the only two Clifford vectors whose square is x.

If $n > 1$, there are infinitely many square roots of a negative number x.

Hyperbolic Space

Upper Half Space Model

$$\mathbf{H}^{n+2} = \mathbb{V}^{n+1} \times \mathbb{R}_{>0}$$

Geodesics are half circles orthogonal to the bounding plane or vertical lines

$$\partial\mathbf{H}^{n+2} = \mathbb{V}^{n+1} \cup \{\infty\}$$

Geodesics can be given by two Clifford vectors (or one ∞)

Hyperbolic Isometries

Clifford Matrices $\mathrm{GL}_2(\Gamma_n)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$\begin{aligned} a, b, c, d &\in \Gamma_n \cup \{0\}\,; \\ ab^\star, cd^\star, c^\star a, d^\star b &\in \mathbb{V}^{n+1}; \\ ad^\star - bc^\star &\in \mathbb{R} \setminus \{0\} \end{aligned}$$

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generated by

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{\star -1} \end{pmatrix}$$

with $t \in \mathbb{V}^{n+1}$ and $a \in \Gamma^n$.

$\mathrm{PSL}_2(\mathcal{C}_n)$

$$\mathrm{PSL}_2\left(\mathcal{C}_n\right) := \mathrm{SL}_2\left(\mathcal{C}_n\right) / \left\{\pm I\right\}$$

Hyperbolic Isometries

$T \in \mathrm{PSL}_2(\mathcal{C}_n)$ acts on $\mathbb{V}^{n+1} \cup \{\infty\} = \partial \mathbf{H}^{n+2}$ by orientation preserving Möbius transformations:

$$T(x) := (ax + b)(cx + d)^{-1}$$

Poincaré Extension

$$\mathrm{Isom}^+\left(\mathbf{H}^{n+2}\right) \cong \mathrm{Möb}^+(n+1) \cong \mathrm{PSL}_2(\mathcal{C}_n)$$

Cross Ratio

Definition

Let $x, y, z, w \in \mathbb{V}^{n+1}$ be pairwise different.
$[x, y, z, w] := (x - z)(x - w)^{-1}(y - w)(y - z)^{-1} \in \Gamma_n \setminus \{0\}$.
Extend the obvious way for one of the variables $= \infty$.

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Let T Möbius transformation for $A \in \mathrm{PSL}_2(\mathcal{C}_n)$. Then

$$[T(x), T(y), T(z), T(w)] = (cz + d)^{\star -1}\,[x, y, z, w]\,(cz + d)^{\star}.$$

Remark

$|[\cdot,\cdot,\cdot,\cdot]|$ and $\Re([\cdot,\cdot,\cdot,\cdot])$ are $\mathrm{PSL}_2(\mathcal{C}_n)$-invariant but *not* cross ratio itself.

Let $s = (s^-, s^+)$ and $t = (t^-, t^+)$ two geodesics given by their endpoints $s^\pm, t^\pm \in \partial\mathbf{H}^{n+2} = \mathbb{V}^{n+1} \cup \{\infty\}$.

Definition

$$\Delta(s, t) := \left[s^-, s^+, t^-, t^+\right]$$

Cross Ratio of Two Geodesics

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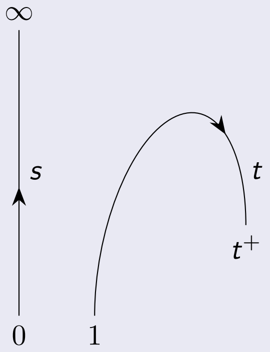
Geometric Meaning



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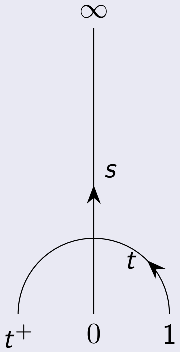

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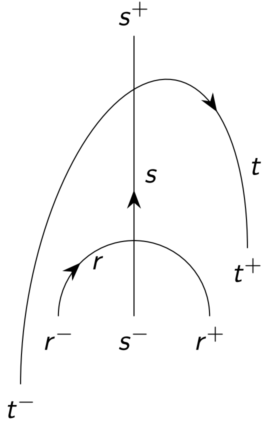
Geodesics s and t

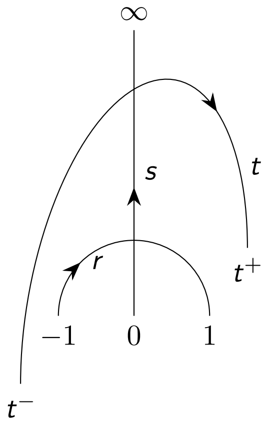
- intersect if $\Delta(s,t) < 0$,
- are orthogonal if $\Delta(s,t) = -1$.

Double Bridge



$r \perp s \perp t$ with pairwise different endpoints

Standard Configuration Double Bridge



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Double Bridge Cross Ratio

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In Standard Configuration

$$\Delta(r,s,t) = \left[\infty, 0, 1, t^+\right] = (0 - t^+)(0-1)^{-1} = t^+ \in \mathbb{V}^{n+1}$$

Constructing p-gons in $\mathbf{H}^{p-1}$

Idea of Construction

- parameters $\{q_1, \ldots, q_{p-3}\} \subset \mathbb{V}^{p-2}$
- correspond to double bridge cross ratio in standard configuration

Gauging of Cross Ratios

Let $\{q_1, \ldots, q_{p-3}\} \subset \mathbb{V}^{p-2}$. Consider

$$\phi_i : x \mapsto \sqrt{-2q_i}^{-1}(x + q_i)(x - q_i)^{-1}\sqrt{-2q_i}.$$

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$$0 \mapsto -1$$

$$\infty \mapsto 1$$

$$-q_i \mapsto 0$$

$$q_i \mapsto \infty$$

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$$\Phi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$$

$$\phi_i^{-1} : x \mapsto \sqrt{-q_i}(1+x)(1-x)^{-1}\sqrt{-q_i}.$$

Deconstructing a p-gon

Let $(S_0, \dots, S_{p-1})$ right-angled p-gon.

Gauged Double Bridge Cross Ratios

$$\tilde{\Delta}_1 := \Delta(S_0, S_1, S_2)$$

$$\vdots$$

$$\tilde{\Delta}_{i+1} := \Delta\left(\Phi_i(S_i), \Phi_i(S_{i+1}), \Phi_i(S_{i+2})\right)$$

$$\vdots$$

$$\tilde{\Delta}_{p-3} := \Delta\left(\Phi_{p-4}(S_{p-4}), \Phi_i(S_{p-3}), \Phi_i(S_{p-2})\right)$$

yields a map

$$\{\text{oriented right-angled polygons in } \mathbf{H}^{p-1}\} \hookrightarrow \left(\mathbb{V}^{p-2}\right)^{p-3}$$

Reconstructing a p-gon

parameters $\{q_1, \ldots, q_{p-3}\} \in \mathbb{V}^{p-2}$

1. fix $S_0 = (-1, 1)$ and $S_1 = (0, \infty)$
2. $S_2 = (-q_1, q_1)$ since $\Delta(S_0, S_1, S_2) = q_1$
3. use gauging: $S_3 = \left(\Phi_1^{-1}(-q_2), \Phi_1^{-1}(q_2)\right)$

   $\vdots$

4. use gauging: $S_{p-2} = \left(\Phi_{p-4}^{-1}(-q_{p-3}), \Phi_{p-4}^{-1}(q_{p-3})\right)$
5. last geodesic S_{p-1} exists and is unique iff $\Delta(S_{p-2}, S_0) \notin \mathbb{R}_{\leq 0}$

Possible Parameters?

Example Pentagon in $\mathbf{H}^4$

Parameters $q_1, q_2 \in \mathbb{V}^3 = \mathbb{R} + \mathbb{R}i + \mathbb{R}j \subset \mathbb{H}$

Choose $q_1 = 2i, q_2 = 2j$

- $S_0 = (-1, 1)$
- $S_1 = (0, \infty)$
- $S_2 =$
- $S_3 =$
- $S_4 =$

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- $S_3 = \left(\Phi^{-1}(-2j), \Phi^{-1}(2j)\right) = \left(\frac{1}{5}(6i - 8j), \frac{1}{5}(6i + 8j)\right)$
- $S_4 =$

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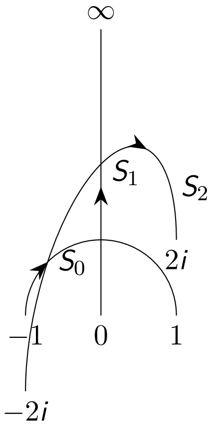
- $S_4 =$ common perpendicular to S_3 and S_0

Sketch



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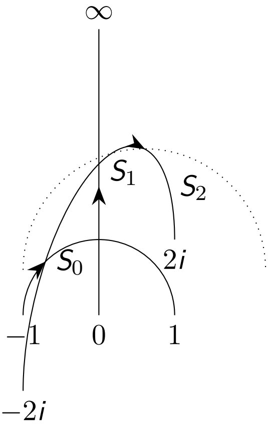

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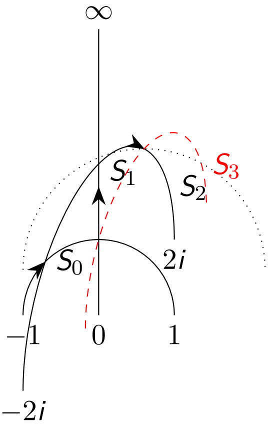

Necessary Condition for Polygons with Full Span

Theorem

If the parameters $q_1, \ldots, q_{p-3}$ give rise to a right-angled polygons such that the intersections are the vertices of a simplex, then

$$\langle 1, q_1, \ldots, q_{p-3} \rangle = \mathbb{V}^{p-2}.$$

François Delgove and Nicolas Retailleau. “Sur la classification des hexagones hyperboliques à angles droits en dimension 5”. eng. In: *Annales de la faculté des sciences de Toulouse Mathématiques* 23.5 (2014), pp. 1049–1061. URL: http://eudml.org/doc/275407.

Edoardo Dotti and Simon T. Drewitz. “On right-angled polygons in hyperbolic space”. In: *Geometriae Dedicata* (2018). ISSN: 1572-9168. DOI: 10.1007/s10711-018-0357-y.